N94-35870

Applying Transfer Matrix Method to the Estimation of the Modal Characteristics of the NASA Mini-Mast Truss

Ji Yao Shen
Dept. of Mechanical Engineering
North Carolina A&T State University
Greensboro, NC 27411

Lawrence W. Taylor, Jr.
Guidance & Control Division
NASA Langley Research Center
Hampton, VA 23665

P. 18

Abstract

It is beneficial to use a distributed parameter model for large space structures, because the approach minimizes the number of model parameters. Holzer's transfer matrix method provides a useful means to simplify and standardize the procedure for solving the system of partial differential equations. Any large space structures can be broken down into sub-structures with simple elastic and dynamical properties. For each single element, such as beam, tether, or rigid body, we can derive the corresponding transfer matrix. Combining these elements' matrices enables the solution of the global system equations. The characteristic equation can then be formed by satisfying the appropriate boundary conditions. Then natural frequencies and mode shapes can be determined by searching the roots of the characteristic equation at frequencies within the range of interest. This paper applies this methodology, and the maximum likelihood estimation method, to refine the modal characteristics of the NASA Mini-Mast Truss by successively matching the theoretical response to the test data of the truss. The method is being applied to more complex configurations.

1. Introduction

Control of flexible spacecraft is best analyzed by representing, in a single set of equations, all of the structural modes and the control system dynamics. Distributed parameter models enable this approach based upon the classical partial differential equation theories. In the early 1960's, the distributed parameter approach was developed simultaneously with the lumped parameter approach [1-3]. With the event of high-speed and large-memory computers the finite element method has been developed much more extensively than the distributed parameter approach. However, the advantages of the distributed parameter approach to control synthesis, parameter estimation and integrated design have been largely neglected by

the technical community. It is the purpose of this paper to show the advantage of employing the distributed parameter approach to large space structures.

Holzer's transfer matrix method provides a useful means to simplify and standardize the procedure for solving the system of partial differential equations. Also, the transfer matrix method enables the construction of a relatively simple mathematical model for complicated structures. It is of great practical value to take advantage of catalogs of the most important transfer matrices readily available. A complex structure, then, only requires combining individual matrices to represent the structure which consists of connected elements. The similarity of this concept to that of the transfer function is particularly useful to control analysis.

The transfer matrix method itself has been a matured method [4], and its power has been shown in several technical areas [5,6]. But, little use of the transfer matrix method has been made for the distributed modeling, parameter estimation and control of large flexible space structures.

In this paper we applied the transfer matrix method, accompanied by the maximum likelihood estimation technique, to estimate the lateral bending characteristics of the NASA Mini-Mast truss (Fig.1.1)[7] by matching the theoretical transient response to test data. The Mini-Mast truss is a ground testbed for the Control-Structure Interaction (CSI) program. The total height of the truss is 20.16 meters, containing 18 deployable bays. Two instrumentation platforms have been installed at Bay10 and Bay18. Mini-Mast has 162 major structural elements. Finite element models of the truss involve thousands of elements. The distributed parameter model of the Mini-Mast truss used in this paper consists of two flexible beam elements and two rigid bodies.

The shear deformation of the truss requires a Timoshenko beam model in order to match the frequencies at higher mode numbers. It is also necessary to extend the simple model by adding the so-called "appendage model" to account for the effects of dynamics of diagonal struts and associated hinge bodies. The method of this paper is shown to be applicable to more complex configurations.

2. Derivation of Transfer Matrix

Holzer's transfer matrix method [4] provides a useful means to simplify and standardize the procedure for solving the partial differential equations. Any large space structures can be broken down into sub-structures with simple elastic and dynamical properties. For each single element, such as beam in bending, rigid body, we can derive the corresponding field matrix and point matrix. Combining these elements' matrices in a required manner, one can calculate the responses, i.e. the solution to the global system equation, by proceeding from one point of the system to the other.

This paper concentrates on the estimation of the lateral bending frequency of the NASA Mini-Mast Truss. The truss is modeled as two successive beam elements with two rigid bodies at the Bay10, and Bay18 (tip of the truss), respectively. To derive the transfer matrix we consider a

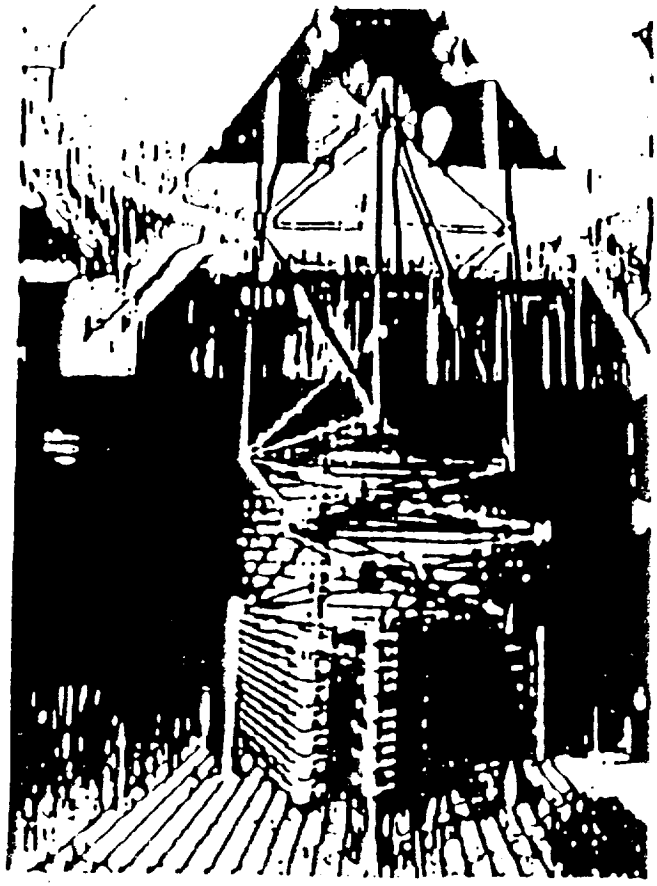

Fig. 1.1 NASA Mini-Mast Truss

"original photo not available"

cascaded beam-body system (Fig.2.1). The jth section consists of a flexible beam whose elastic and dynamical properties will be described by a field matrix, and a rigid body whose dynamical property is presented by a point matrix. When necessary, we will designate any corresponding quantities to the left and right of a rigid body by superscripts L and R.

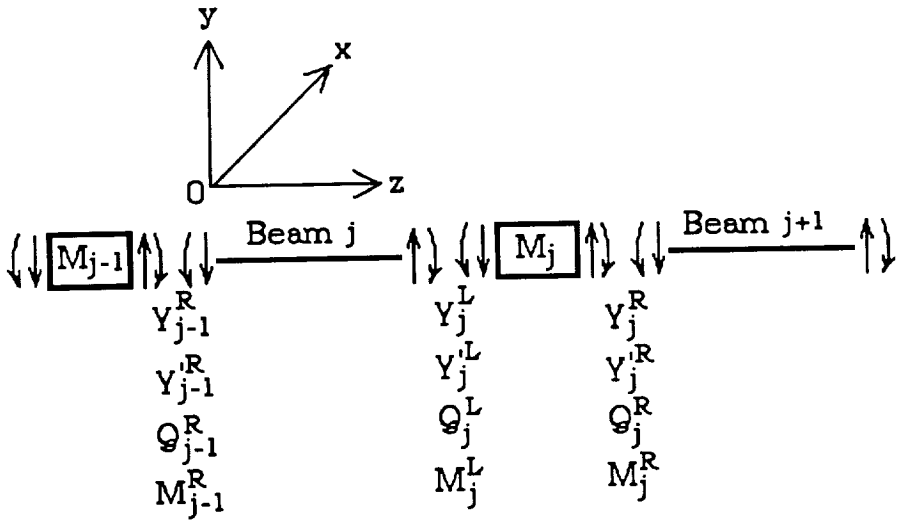

Fig. 2.1 A Cascaded Beam-Body System

The lateral bending of the beam is represented by the Bernoulli-Euler beam equation,

$$\frac{\partial^4 y}{\partial z^4} + \frac{1}{a^2}\frac{\partial^2 y}{\partial t^2} = 0 \qquad (2.1)$$

where, y(z,t) is the lateral displacement, and $a^2=k/m$ where k=EI bending stiffness and m=ρA mass per length of the beam. By separation of variables y(z,t)=Y(z)T(t), we have two ordinary differential equations in Y(z) and T(t),

$$Y''''(z) - \beta^4 Y(z) = 0 \qquad (2.2)$$

and

$$\ddot{T}(t) + \omega^2 T(t) = 0 \qquad (2.3)$$

where, $\beta^4=\omega^2/a^2$. The solution to the Eq.(2.2) has the form,

$$Y(z) = A\sin\beta z + B\cos\beta z + C\sinh\beta z + D\cosh\beta z \qquad (2.4)$$

At the left end of the beam (z=0), the displacement Y(0), slope Y'(0), shear Q(0), and bending moment M(0) will be

$$Y^R_{j-1} = B_j + D_j$$
$$Y'^R_{j-1} = A_j\beta_j + C_j\beta_j$$
$$Q^R_{j-1} = -k_jY_j'''(0) = A_jk_j\beta_j^3 - C_jk_j\beta_j^3$$
$$M^R_{j-1} = -k_jY_j''(0) = B_jk_j\beta_j^2 - D_jk_j\beta_j^2$$

or written in matrix form as,

$$\begin{Bmatrix} Y \\ Y' \\ Q \\ M \end{Bmatrix}^R_{j-1} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ \beta & 0 & \beta & 0 \\ k\beta^3 & 0 & -k\beta^3 & 0 \\ 0 & k\beta^2 & 0 & -k\beta^2 \end{bmatrix}_j \begin{Bmatrix} A \\ B \\ C \\ D \end{Bmatrix}_j$$

Thus,

$$\begin{Bmatrix} A \\ B \\ C \\ D \end{Bmatrix}_j = \begin{bmatrix} 0 & 1 & 0 & 1 \\ \beta & 0 & \beta & 0 \\ k\beta^3 & 0 & -k\beta^3 & 0 \\ 0 & k\beta^2 & 0 & -k\beta^2 \end{bmatrix}^{-1}_j \begin{Bmatrix} Y \\ Y' \\ Q \\ M \end{Bmatrix}^R_{j-1} = \begin{bmatrix} 0 & \frac{1}{2\beta} & \frac{1}{2k\beta^3} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2k\beta^2} \\ 0 & \frac{1}{2\beta} & -\frac{1}{2k\beta^3} & 0 \\ \frac{1}{2} & 0 & 0 & -\frac{1}{2k\beta^2} \end{bmatrix}_j \begin{Bmatrix} Y \\ Y' \\ Q \\ M \end{Bmatrix}^R_{j-1} \quad (2.5)$$

Similarly, at the right end of the beam ($z=L_j$), the corresponding quantities are,

$$Y^L_j = A_j\sin\beta_jL_j + B_j\cos\beta_jL_j + C_j\sinh\beta_jL_j + D_j\cosh\beta_jL_j$$
$$Y'^L_j = A_j\beta_j\cos\beta_jL_j - B_j\beta_j\sin\beta_jL_j + C_j\beta_j\cosh\beta_jL_j + D_j\beta_j\sinh\beta_jL_j$$
$$Q^L_j = kY_j'''(L) = -A_jk_j\beta_j^3\cos\beta_jL_j + B_jk_j\beta_j^3\sin\beta_jL_j + C_jk_j\beta_j^3\cosh\beta_jL_j + D_jk_j\beta_j^3\sinh\beta_jL_j$$
$$M^L_j = kY_j''(L) = -A_jk_j\beta_j^2\sin\beta_jL_j - B_jk_j\beta_j^2\cos\beta_jL_j + C_jk_j\beta_j^2\sinh\beta_jL_j + D_jk_j\beta_j^2\cosh\beta_jL_j$$

or written in matrix form as,

$$\begin{Bmatrix} Y \\ Y' \\ Q \\ M \end{Bmatrix}^L_j = \begin{bmatrix} \sin\beta L & \cos\beta L & \sinh\beta L & \cosh\beta L \\ \beta\cos\beta L & -\beta\sin\beta L & \beta\cosh\beta L & \beta\sinh\beta L \\ -k\beta^3\cos\beta L & k\beta^3\sin\beta L & k\beta^3\cosh\beta L & k\beta^3\sinh\beta L \\ -k\beta^2\sin\beta L & -k\beta^2\cos\beta L & k\beta^2\sinh\beta L & k\beta^2\cosh\beta L \end{bmatrix}_j \begin{Bmatrix} A \\ B \\ C \\ D \end{Bmatrix}_j \quad (2.6)$$

Substituting Eq.(2.5) into Eq.(2.6) we obtain

$$\begin{Bmatrix} Y \\ Y' \\ Q \\ M \end{Bmatrix}_j^L = [FM]_j \begin{Bmatrix} Y \\ Y' \\ Q \\ M \end{Bmatrix}_{j-1}^R \tag{2.7}$$

where, the field matrix of the jth beam element,

$$[FM]_j = \begin{bmatrix} \frac{1}{2}(\cos\beta L+\cosh\beta L) & \frac{1}{2\beta}(\sin\beta L+\sinh\beta L) & \frac{1}{2k\beta^3}(\sin\beta L-\sinh\beta L) & \frac{1}{2k\beta^2}(\cos\beta L-\cosh\beta L) \\ -\frac{\beta}{2}(\sin\beta L-\sinh\beta L) & \frac{1}{2}(\cos\beta L+\cosh\beta L) & \frac{1}{2k\beta^2}(\cos\beta L-\cosh\beta L) & -\frac{1}{2k\beta}(\sin\beta L+\sinh\beta L) \\ \frac{1}{2}k\beta^3(\sin\beta L+\sinh\beta L) & -\frac{1}{2}k\beta^2(\cos\beta L-\cosh\beta L) & -\frac{1}{2}(\cos\beta L+\cosh\beta L) & \frac{\beta}{2}(\sin\beta L-\sinh\beta L) \\ -\frac{1}{2}k\beta^2(\cos\beta L-\cosh\beta L) & -\frac{1}{2}k\beta(\sin\beta L-\sinh\beta L) & -\frac{1}{2\beta}(\sin\beta L+\sinh\beta L) & -\frac{1}{2}(\cos\beta L+\cosh\beta L) \end{bmatrix}_j$$

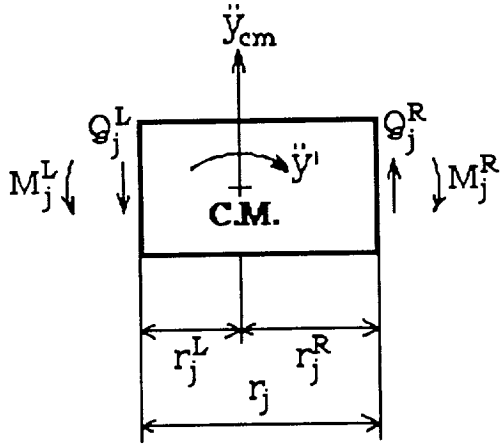

Fig. 2.2 Free Diagram of a Rigid Body

Next, let us consider the jth body (Fig.2.2). The translational and rotational motions of the body can be described by the following equations,

$$m_j \ddot{y}_{cm} = Q_j^R - Q_j^L \tag{2.8}$$

$$I_j \ddot{y}'^L = M_j^R - M_j^L - Q_j^R r_j^R - Q_j^L r_j^L \tag{2.9}$$

For homogeneous motions, Eqs.(2.8) and (2.9) can be written as,

$$Q_j^R = Q_j^L - m_j\omega^2 Y_{cm} \tag{2.10}$$

$$M_j^R = M_j^L + Q_j^R r_j^R + Q_j^L r_j^L - I_j\omega^2 Y'^L_j \tag{2.11}$$

But, the displacement of the center of mass, Y_{cm}, is related to Yj^R and Yj^L by,

$$Y_j^R = Y_{cm} - r_j^R \, Y'^R_j \tag{2.12}$$

$$Y_j^L = Y_{cm} + r_j^L \, Y'^L_j \tag{2.13}$$

From Eq.(2.13), we see that

$$Y_{cm} = Y_j^L - r_j^L \, Y'^L_j \tag{2.14}$$

To keep the compatibility of the deflection, it must be that

$$Y'^R_j = Y'^L_j \tag{2.15}$$

Substitution of Eqs.(2.14) and (2.15) into Eq.(2.12) gives

$$Y_j^R = Y_j^L - r_j \, Y'^L_j \tag{2.16}$$

where, $r_j = r_j^L + r_j^R$.

Substituting Eq.(2.14) into Eq.(2.10) we obtain

$$Q_j^R = Q_j^L - m_j\omega^2 (Y_j^L - r_j^L \, Y'^L_j) \tag{2.17}$$

Substituting Eq.(2.17) into Eq.(2.11) we can derive

$$M_j^R = M_j^L + r_j \, Q_j^L - m_j\omega^2 r_j^R Y_j^L - (I_j - m_j r_j^L r_j^R)\omega^2 Y'^L_j \tag{2.18}$$

Collecting Eqs.(2.16), (2.15), (2.17) and (2.18), and writing in the matrix form we obtain

$$\begin{Bmatrix} Y \\ Y' \\ Q \\ M \end{Bmatrix}^R_j = [PM]_j \begin{Bmatrix} Y \\ Y' \\ Q \\ M \end{Bmatrix}^L_j \tag{2.19}$$

where, the point matrix of the rigid body element,

$$[PM]_j = \begin{bmatrix} 1 & -r_j & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -m_j\omega^2 & m_j\omega^2 r_j^L & 1 & 0 \\ -m_j\omega^2 r_j^R & -(I_j - m_j r_j^R r_j^L)\omega^2 & r_j & 1 \end{bmatrix}$$

Substituting Eq.(2.7) into Eq.(2.19) we obtain

$$\begin{Bmatrix} Y \\ Y' \\ Q \\ M \end{Bmatrix}^{R} = [\Phi]_{j,\,j-1} \begin{Bmatrix} Y \\ Y' \\ Q \\ M \end{Bmatrix}^{R} \qquad (2.20)$$

where, the transfer matrix of the jth section,

$$[\Phi]_{j,\,j-1} = [PM]_j [FM]_j = \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} & \Phi_{14} \\ \Phi_{21} & \Phi_{22} & \Phi_{23} & \Phi_{24} \\ \Phi_{31} & \Phi_{32} & \Phi_{33} & \Phi_{34} \\ \Phi_{41} & \Phi_{42} & \Phi_{43} & \Phi_{44} \end{bmatrix}_{j,\,j-1}$$

The elements of the transfer matrix are listed below:

$$\Phi_{11} = \frac{1}{2}(\cos\beta L + \cosh\beta L) + \frac{1}{2}\beta r(\sin\beta L - \sinh\beta L)$$
$$\Phi_{12} = \frac{1}{2\beta}(\sin\beta L + \sinh\beta L) - \frac{r}{2}(\cos\beta L + \cosh\beta L)$$
$$\Phi_{13} = \frac{1}{2k\beta^3}(\sin\beta L - \sinh\beta L) - \frac{r}{2k\beta^2}(\cos\beta L - \cosh\beta L)$$
$$\Phi_{14} = \frac{1}{2k\beta^2}(\cos\beta L - \cosh\beta L) + \frac{r}{2k\beta}(\sin\beta L + \sinh\beta L)$$

$$\Phi_{21} = -\frac{\beta}{2}(\sin\beta L - \sinh\beta L)$$
$$\Phi_{22} = \frac{1}{2}(\cos\beta L + \cosh\beta L)$$
$$\Phi_{23} = \frac{1}{2k\beta^2}(\cos\beta L - \cosh\beta L)$$
$$\Phi_{24} = -\frac{1}{2k\beta}(\sin\beta L + \sinh\beta L)$$

$$\Phi_{31} = -\frac{1}{2}m\omega^2(\cos\beta L + \cosh\beta L) - \frac{1}{2}\beta m\omega^2 r^L(\sin\beta L - \sinh\beta L) + \frac{1}{2}k\beta^3(\sin\beta L + \sinh\beta L)$$
$$\Phi_{32} = -\frac{1}{2\beta}m\omega^2(\sin\beta L + \sinh\beta L) + \frac{1}{2}m\omega^2 r^L(\cos\beta L + \cosh\beta L) - \frac{1}{2}k\beta^2(\cos\beta L - \cosh\beta L)$$
$$\Phi_{33} = -\frac{m\omega^2}{2k\beta^3}(\sin\beta L - \sinh\beta L) + \frac{m\omega^2 r^L}{2k\beta^2}(\cos\beta L - \cosh\beta L) - \frac{1}{2}(\cos\beta L + \cosh\beta L)$$
$$\Phi_{34} = -\frac{m\omega^2}{2k\beta^2}(\cos\beta L - \cosh\beta L) - \frac{m\omega^2 r^L}{2k\beta}(\sin\beta L + \sinh\beta L) + \frac{\beta}{2}(\sin\beta L - \sinh\beta L)$$

$$\Phi_{41} = -\frac{1}{2}m\omega^2 r^R(\cos\beta L + \cosh\beta L) + \frac{\beta}{2}(I - m r^R r^L)\omega^2(\sin\beta L - \sinh\beta L)$$
$$+ \frac{1}{2}k\beta^3 r(\sin\beta L + \sinh\beta L) - \frac{1}{2}k\beta^2(\cos\beta L - \cosh\beta L)$$

$$\Phi_{42} = -\frac{1}{2\beta}m\omega^2 r^R(\sin\beta L + \sinh\beta L) - \frac{1}{2}(I - mr^R r^L)\omega^2(\cos\beta L + \cosh\beta L)$$
$$-\frac{1}{2}k\beta^2 r(\cos\beta L - \cosh\beta L) - \frac{1}{2}k\beta(\sin\beta L - \sinh\beta L)$$
$$\Phi_{43} = -\frac{1}{2k\beta^3}m\omega^2 r^R(\sin\beta L - \sinh\beta L) - \frac{1}{2k\beta^2}(I - mr^R r^L)\omega^2(\cos\beta L - \cosh\beta L)$$
$$-\frac{r}{2}(\cos\beta L + \cosh\beta L) - \frac{1}{2\beta}(\sin\beta L + \sinh\beta L)$$
$$\Phi_{44} = -\frac{1}{2k\beta^2}m\omega^2 r^R(\cos\beta L - \cosh\beta L) + \frac{1}{2k\beta}(I - mr^R r^L)\omega^2(\sin\beta L + \sinh\beta L)$$
$$+\frac{1}{2}\beta r(\sin\beta L - \sinh\beta L) - \frac{1}{2}(\cos\beta L + \cosh\beta L)$$

3. Characteristic Equation: Eigenvalue and Eigenfunction

After establishing the equation of motion of a global system by combining the transfer functions of all necessary elements, the natural frequency and mode shape function can be solved by satisfying the appropriate boundary conditions.

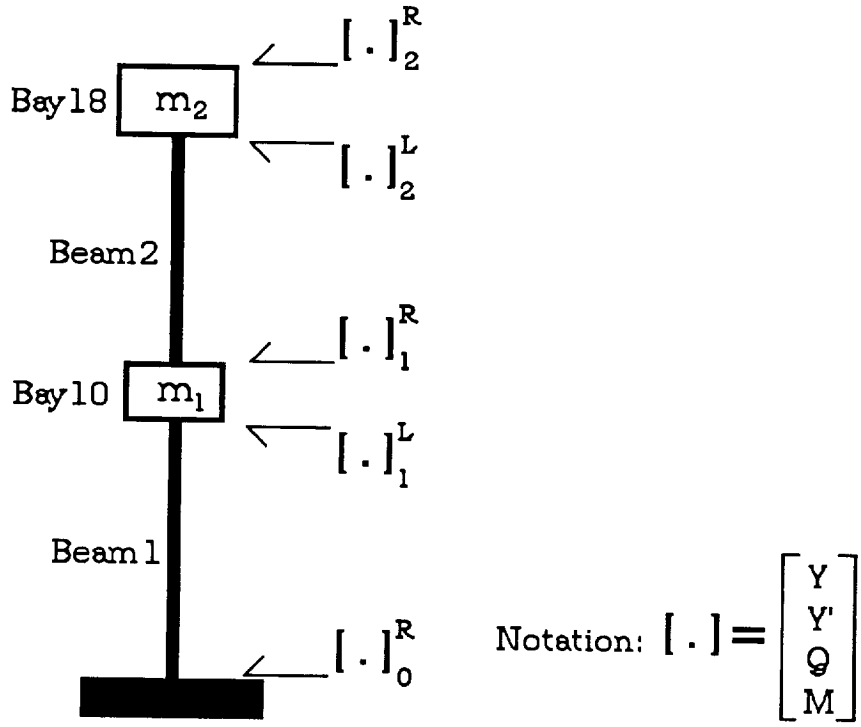

Fig. 3.1 The Mathematical Model for NASA Mini-Mast

As a mathematical model of the NASA Mini-Mast truss, it consists of two successive beam elements with two rigid bodies at the Bay10 and Bay18 (Fig.3.1). Using the transfer function in Eq.(2.20) we can see

$$\begin{Bmatrix} Y \\ Y' \\ Q \\ M \end{Bmatrix}_2^R = [\Phi]_{2,1}[\Phi]_{1,0} \begin{Bmatrix} Y \\ Y' \\ Q \\ M \end{Bmatrix}_0^R = [\Phi]_{2,0} \begin{Bmatrix} Y \\ Y' \\ Q \\ M \end{Bmatrix}_0^R \tag{3.1}$$

Now let us consider the boundary conditions. At the fixed end we have

$$Y_0 = Y'_0 = 0 \tag{3.2}$$

at the free end,

$$Q_2 = M_2 = 0 \tag{3.3}$$

Applying the BC.'s to the Eq.(3.1) we get

$$\begin{Bmatrix} Y \\ Y' \\ 0 \\ 0 \end{Bmatrix}_2^R = [\Phi]_{2,0} \begin{Bmatrix} 0 \\ 0 \\ Q \\ M \end{Bmatrix}_0^R \tag{3.4}$$

Rearranging the state vector we will have

$$[A] \begin{Bmatrix} Y_2 \\ Y'_2 \\ Q_0 \\ M_0 \end{Bmatrix}^R = [0] \tag{3.5}$$

The condition for Eq.(3.5) having non-trivial solution is that the determinant of the coefficient matrix equals to zero, that is,

$$\text{Det}[A] = \text{Det} \begin{bmatrix} -1 & 0 & \Phi_{13} & \Phi_{14} \\ 0 & -1 & \Phi_{23} & \Phi_{24} \\ 0 & 0 & \Phi_{33} & \Phi_{34} \\ 0 & 0 & \Phi_{43} & \Phi_{44} \end{bmatrix} = 0 \tag{3.6}$$

where, $\Phi_{i,j}$'s (i=1 to 4, j=3,4) are the elements of the transfer matrix $[\Phi]_{2,0}$. Eq.(3.6) is the so-called characteristic equation. Expanding the determinant in Eq.(3.6) we can rewrite the characteristic equation as

$$\Phi_{33}\,\Phi_{44} - \Phi_{34}\,\Phi_{43} = 0 \tag{3.7}$$

Solving for the roots of the characteristic equation, Eq.(3.7), we can get the eigenvalue's β's. To verify the theoretical derivation we have deduced the characteristic equations for two simple examples from the foregoing characteristic equation (Eq.3.7) as follows:
For a cantilevered beam:

$$\cos\beta L \cdot \cosh\beta L = -1$$

For a cantilevered beam with a tip body:

$$1 + \cos\beta L \cdot \cosh\beta L = -\frac{W}{W_b}\beta L\,(\cos\beta L \sinh\beta L - \sin\beta L \cosh\beta L)$$

where W_b is the weight of the beam; W is the weight of the body. These results are identical with those given in commonly used textbooks [8].

To establish the mode shape functions we must solve the Eqs.(3.5) and (2.19) simultaneously, that is,

$$\begin{bmatrix} -1 & 0 & \Phi_{13} & \Phi_{14} \\ 0 & -1 & \Phi_{23} & \Phi_{24} \\ 0 & 0 & \Phi_{33} & \Phi_{34} \\ 0 & 0 & \Phi_{43} & \Phi_{44} \end{bmatrix} \begin{Bmatrix} Y_2 \\ Y'_2 \\ Q_0 \\ M_0 \end{Bmatrix}^R = 0 \qquad (3.8)$$

and

$$\begin{Bmatrix} Y \\ Y' \\ Q \\ M \end{Bmatrix}^R_1 = [PM]_1 \begin{Bmatrix} Y \\ Y' \\ Q \\ M \end{Bmatrix}^L_1 \qquad (3.9)$$

The later equation represents the compatible conditions of the deflections, forces, and moments between the two sides of the Body1. Because of the translational and rotary inertias of the Body1, the shear and bending moment have jumps at the connection point of the two beam elements, while the deflection functions are still continuous.

According to the solution function, Eq.(2.4), we can express the state vectors in Eqs.(3.8) and (3.9) in terms of the coefficients A_j, B_j, C_j and D_j (j=1,2) by using the following relations:

$$\begin{aligned}
Q_0^R &= A_1k_1\beta_1^3 - C_1k_1\beta_1^3 \\
M_0^R &= B_1k_1\beta_1^2 - D_1k_1\beta_1^2 \\
\\
Y_1^L &= A_1\sin\beta_1L_1 + B_1\cos\beta_1L_1 + C_1\sinh\beta_1L_1 + D_1\cosh\beta_1L_1 \\
Y'^L_1 &= A_1\beta_1\cos\beta_1L_1 - B_1\beta_1\sin\beta_1L_1 + C_1\beta_1\cosh\beta_1L_1 + D_1\beta_1\sinh\beta_1L_1 \\
Q_1^L &= -A_1k_1\beta_1^3\cos\beta_1L_1 + B_1k_1\beta_1^3\sin\beta_1L_1 + C_1k_1\beta_1^3\cosh\beta_1L_1 + D_1k_1\beta_1^3\sinh\beta_1L_1 \\
M_1^L &= -A_1k_1\beta_1^2\sin\beta_1L_1 - B_1k_1\beta_1^2\cos\beta_1L_1 + C_1k_1\beta_1^2\sinh\beta_1L_1 + D_1k_1\beta_1^2\cosh\beta_1L_1 \\
\\
Y_1^R &= B_2 + D_2 \\
Y'^R_1 &= A_2\beta_2 + C_2\beta_2
\end{aligned} \qquad (3.10)$$

$$Q_1^R = A_2k_2\beta_2^3 - C_2k_2\beta_2^3$$
$$M_1^R = B_2k_2\beta_2^2 - D_2k_2\beta_2^2$$

$$Y_2^R \equiv Y_2^L = A_2\sin\beta_2L_2 + B_2\cos\beta_2L_2 + C_2\sinh\beta_2L_2 + D_2\cosh\beta_2L_2$$
$$Y'^R_2 = Y'^L_2 = A_2\beta_2\cos\beta_2L_2 - B_2\beta_2\sin\beta_2L_2 + C_2\beta_2\cosh\beta_2L_2 + D_2\beta_2\sinh\beta_2L_2$$

Substituting the quantities in Eq.(3.10) into Eqs.(3.8) and (3.9) we obtain a matrix equation in the coefficients A_j, B_j, C_j and D_j (j=1,2),

$$\begin{bmatrix} k_1\beta_1^3\Phi_{13} & k_1\beta_1^2\Phi_{14} & -k_1\beta_1^3\Phi_{13} & -k_1\beta_1^2\Phi_{14} & \alpha_{15} & \alpha_{16} & \alpha_{17} & \alpha_{18} \\ k_1\beta_1^3\Phi_{23} & k_1\beta_1^2\Phi_{24} & -k_1\beta_1^3\Phi_{23} & -k_1\beta_1^2\Phi_{24} & \alpha_{25} & \alpha_{26} & \alpha_{27} & \alpha_{28} \\ k_1\beta_1^3\Phi_{33} & k_1\beta_1^2\Phi_{34} & -k_1\beta_1^3\Phi_{33} & -k_1\beta_1^2\Phi_{34} & 0 & 0 & 0 & 0 \\ k_1\beta_1^3\Phi_{43} & k_1\beta_1^2\Phi_{44} & -k_1\beta_1^3\Phi_{43} & -k_1\beta_1^2\Phi_{44} & 0 & 0 & 0 & 0 \\ \alpha_{51} & \alpha_{52} & \alpha_{53} & \alpha_{54} & 0 & -1 & 0 & -1 \\ \alpha_{61} & \alpha_{62} & \alpha_{63} & \alpha_{64} & -\beta_2 & 0 & -\beta_2 & 0 \\ \alpha_{71} & \alpha_{72} & \alpha_{73} & \alpha_{74} & -k_2\beta_2^3 & 0 & k_2\beta_2^3 & 0 \\ \alpha_{81} & \alpha_{82} & \alpha_{83} & \alpha_{84} & 0 & -k_2\beta_2^2 & 0 & k_2\beta_2^2 \end{bmatrix} \begin{Bmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \\ A_2 \\ B_2 \\ C_2 \\ D_2 \end{Bmatrix} = [0] \quad (3.11)$$

The elements expressed in α_{ij}'s in coefficient matrix are listed below:

$\alpha_{15} = -\sin\beta_2L_2$
$\alpha_{16} = -\cos\beta_2L_2$
$\alpha_{17} = -\sinh\beta_2L_2$
$\alpha_{18} = -\cosh\beta_2L_2$

$\alpha_{25} = -\beta_2\cos\beta_2L_2$
$\alpha_{26} = \beta_2\sin\beta_2L_2$
$\alpha_{27} = -\beta_2\cosh\beta_2L_2$
$\alpha_{28} = -\beta_2\sinh\beta_2L_2$

$\alpha_{51} = \sin\beta_1L_1 - r_1\beta_1\cos\beta_1L_1$
$\alpha_{52} = \cos\beta_1L_1 + r_1\beta_1\sin\beta_1L_1$
$\alpha_{53} = \sinh\beta_1L_1 - r_1\beta_1\cosh\beta_1L_1$
$\alpha_{54} = \cosh\beta_1L_1 - r_1\beta_1\sinh\beta_1L_1$

$\alpha_{61} = \beta_1\cos\beta_1L_1$
$\alpha_{62} = -\beta_1\sin\beta_1L_1$
$\alpha_{63} = \beta_1\cosh\beta_1L_1$
$\alpha_{64} = \beta_1\sinh\beta_1L_1$

$$\alpha_{71} = - m_1\omega^2 \sin\beta_1 L_1 + m_1\omega^2 r_1^L \beta_1 \cos\beta_1 L_1 - k_1\beta_1^3 \cos\beta_1 L_1$$
$$\alpha_{72} = - m_1\omega^2 \cos\beta_1 L_1 - m_1\omega^2 r_1^L \beta_1 \sin\beta_1 L_1 + k_1\beta_1^3 \sin\beta_1 L_1$$
$$\alpha_{73} = - m_1\omega^2 \sinh\beta_1 L_1 + m_1\omega^2 r_1^L \beta_1 \cosh\beta_1 L_1 + k_1\beta_1^3 \cosh\beta_1 L_1$$
$$\alpha_{74} = - m_1\omega^2 \cosh\beta_1 L_1 + m_1\omega^2 r_1^L \beta_1 \sinh\beta_1 L_1 + k_1\beta_1^3 \sinh\beta_1 L_1$$

$$\alpha_{81} = - m_1\omega^2 r_1^R \sin\beta_1 L_1 - (I_1 - m_1 r_1^R r_1^L)\omega^2 \beta_1 \cos\beta_1 L_1 - r_1 k_1 \beta_1^3 \cos\beta_1 L_1 - k_1\beta_1^2 \sin\beta_1 L_1$$
$$\alpha_{82} = - m_1\omega^2 r_1^R \cos\beta_1 L_1 + (I_1 - m_1 r_1^R r_1^L)\omega^2 \beta_1 \sin\beta_1 L_1 + r_1 k_1 \beta_1^3 \sin\beta_1 L_1 - k_1\beta_1^2 \cos\beta_1 L_1$$
$$\alpha_{83} = - m_1\omega^2 r_1^R \sinh\beta_1 L_1 - (I_1 - m_1 r_1^R r_1^L)\omega^2 \beta_1 \cosh\beta_1 L_1 + r_1 k_1 \beta_1^3 \cosh\beta_1 L_1 + k_1\beta_1^2 \sinh\beta_1 L_1$$
$$\alpha_{84} = - m_1\omega^2 r_1^R \cosh\beta_1 L_1 - (I_1 - m_1 r_1^R r_1^L)\omega^2 \beta_1 \sinh\beta_1 L_1 + r_1 k_1 \beta_1^3 \sinh\beta_1 L_1 + k_1\beta_1^2 \cosh\beta_1 L_1$$

The solution to the Eq.(3.11) will give infinite sets of coefficients corresponding to each order of eigenvalue, which are usually called the modal participant coefficients. Assume that the solution is normalized with respect to D_2, that is, $D_2=1$. Then we will have a specific set of coefficients corresponding to an eigenvalue β_{ji}, which is now assumed in the form of

$$\begin{array}{llll} A_1 = c_{1_i}, & B_1 = c_{2_i}, & C_1 = c_{3_i}, & D_1 = c_{4_i} \quad \text{For Beam1} \\ A_2 = c_{5_i}, & B_2 = c_{6_i}, & C_2 = c_{7_i}, & D_2 = 1 \quad \text{For Beam2} \end{array} \tag{3.12}$$

Substituting the coefficients in Eq.(3.12) into the solution equation Eq.(2.4), we obtain the eigenfunctions, or the mode shape functions,

$$Y_i(z) = \begin{cases} c_1 \sin\beta_1 z + c_2 \cos\beta_1 z + c_3 \sinh\beta_1 z + c_4 \cosh\beta_1 z & \text{For Beam1} \\ c_5 \sin\beta_2 z + c_6 \cos\beta_2 z + c_7 \sinh\beta_2 z + \cosh\beta_2 z & \text{For Beam2} \end{cases} \tag{3.13}$$

4. Theoretical Response

When the proportional damping is taken into account the Bernoulli-Euler beam equation will be

$$m\frac{\partial^2 y}{\partial t^2} + c\frac{\partial y}{\partial t} + k\frac{\partial^4 y}{\partial z^4} = 0 \tag{4.1}$$

where, c is a damping constant of proportionality which is assumed as c=2bm. After separation of variables and introducing the generalized coordinates $T_i(t)$, we obtain

$$\sum_i \ddot{T}_i(t)\int_0^L [mY_iY_j]\,dz + \sum_i \dot{T}_i\int_0^L [cY_iY_j]\,dz + \sum_i T_i\int_0^L [kY_iY_j]\,dz = 0$$

which yields all zeros except for the one term in each when i=j according to the orthogonal property of the eigenfunctions, that is,

$$m_i \ddot{T}_i(t) + c_i \dot{T}_i(t) + k_i T_i(t) = 0 \qquad (4.2)$$

where,

$$m_i = \int_0^L m Y_i^2 \, dz \qquad \text{generalized mass}$$

$$k_i = \int_0^L k Y_i^2 \, dz \qquad \text{generalized stiffness}$$

$$c_i = \int_0^L c Y_i^2 \, dz = 2bm_i \qquad \text{generalized damping}$$

Expressing Eq.(4.2) in modal form we obtain

$$\ddot{T}_i + 2\xi_i\omega_i \dot{T}_i + \omega_i^2 T_i = 0 \qquad (4.3)$$

where, $2\xi_i \omega_i = c_i/m_i$ and $\omega_i^2 = k_i/m_i$. The solution to Eq.(4.3) is now

$$T_i(t) = e^{-\xi\omega t} (A_i \cos \omega_d t + B_i \sin \omega_d t) \qquad (4.4)$$

where, A_i and B_i are the coefficients dependent on the initial conditions. By superposition, the solution to the Eq.(4.1) can be written as

$$y(z, t) = \sum_i Y_i(z) e^{-\xi\omega t} (A_i \cos \omega_d t + B_i \sin \omega_d t) \qquad (4.5)$$

where, the eigenfunction $Y_i(z)$ has been derived in Eq.(3.13). Recall that we have defined that

$$a^2=k/m, \qquad 2b=c/m, \qquad \text{and} \qquad \beta^4=\omega^2/a^2 \qquad (4.6)$$

Then the damping ratio ξ_i and the damped natural frequency ω_{di} can be expressed in terms of the parameters a and b,

$$\xi_i = \frac{c_i}{2m_i\omega_i} = \frac{b}{a\beta_i^2}, \quad \text{and} \quad \omega_{d_i} = \omega_i \sqrt{1 - \xi_i^2} = \sqrt{(a\beta_i^2)^2 - b^2} \qquad (4.7)$$

By superposition, finally, the solution to the Eq.(4.1) can also be written in terms of the parameters a and b,

$$y(z,t) = \sum_i Y_i(z)\, e^{-bt} \left(A_i \cos t\sqrt{(a\beta_i^2)^2 - b^2} + B_i \sin t\sqrt{(a\beta_i^2)^2 - b^2} \right) \qquad (4.8)$$

5. Estimation of Modal Characteristics of NASA Mini-Mast

The maximum likelihood estimation method (MLE) is used to estimate the modal characteristics of the NASA Mini-Mast truss. The iterative formula for the MLE estimator has been derived in Ref.[9],

$$\hat{\theta} = \theta_0 + \left[\sum_{j=1}^{m} (\nabla_\theta y_j)^T \mathbf{R}^{-1} (\nabla_\theta y_j) \right]^{-1} \left[\sum_{j=1}^{m} (\nabla_\theta y_j)^T \mathbf{R}^{-1} (y_j - \bar{y}_{j0}) \right] \qquad (5.1)$$

where,

$\bar{y}_0$ nominal response calculated by using θ_0

θ_0 nominal θ vector

$\nabla_\theta y_j$ gradient of y with respect to θ

$\mathbf{R}$ covariance of the measurement noise

The unknown parameter vector θ will be defined as

$$\theta = [a,\ b,\ A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_n]^T \qquad (5.2)$$

Because we have got the closed-form solution y(z,t) (Eq.4.8), the gradient $\nabla_\theta y_j$ can be easily obtained by directly taking the derivatives of y with respect to the unknowns. The closed-form expressions of the gradients have been derived,

$$\frac{\partial y}{\partial a}(z,t) = \sum_i Y_i(z)\, e^{-bt} \frac{\beta_i^2\, a\, t}{\sqrt{(a\beta_i^2)^2 - b^2}} \left[-A_i \sin t\sqrt{(a\beta_i^2)^2 - b^2} + B_i \cos t\sqrt{(a\beta_i^2)^2 - b^2} \right]$$

$$\frac{\partial y}{\partial b}(z,t) = \sum_i Y_i(z)\, t\, e^{-bt} \left\{ -A_i \cos t\sqrt{(a\beta_i^2)^2 - b^2} - B_i \sin t\sqrt{(a\beta_i^2)^2 - b^2} \right.$$
$$\left. + \frac{b}{\sqrt{(a\beta_i^2)^2 - b^2}} \left[A_i \sin t\sqrt{(a\beta_i^2)^2 - b^2} - B_i \cos t\sqrt{(a\beta_i^2)^2 - b^2} \right] \right\}$$

$$\frac{\partial y}{\partial A_i}(z,t) = Y_i(z)\, e^{-bt} \cos t\sqrt{(a\beta_i^2)^2 - b^2}$$

$$\frac{\partial y}{\partial B_i}(z,t) = Y_i(z)\, e^{-bt} \sin t\sqrt{(a\beta_i^2)^2 - b^2} \qquad (5.3)$$

Now we use Eq.(5.1) iteratively, considering y_j as the measurements on a certain location of the beam at each time instant, and $\overline{y}j_0$ as the iterative response values calculated by using the updated θ_0, at the same location and instant. When the innovation of the unknown parameter vector reaches the required criterion we may obtain the estimate $\hat{\theta}$ of θ. All the modal properties are related to the parameters a and b (Eq.4.7), thus we can obtain the modal properties as long as these unknown parameters are determined.

The test data is contained in Ref.[10], which was measured by one displacement sensor installed at Bay18, mounted parallel to the flat face on the corner joints of the structure and positioned to measure deflections normal to the face.

Table 5.1 shows the estimated frequencies which are compared with those obtained from Finite Element Analysis (FEA) and an Eigensystem Realization Algorithm (ERA) [11]. Fig.5.1 shows that the reconstructed response obtained from the estimated parameters and the measured response have a reasonably good fit.

6. Concluding Remarks

This paper has demonstrated the principles for applying a transfer matrix method to the parameter estimation of large space structures. The transfer matrix for the system with flexible beam elements and rigid bodies has been derived. The procedure for establishing natural frequency and mode shape has been described in detail. Maximum likelihood estimation method has served to conduct the parameter estimation. Comparing with the finite element model, the decrease in the number of unknown parameters by the present method is significant. The calculation, therefore, becomes highly efficient. The estimated results are compatible with those obtained by other traditional methods.

Further research is needed to formulate a more general method for more complicated structures. Some problems require coordinate transformation for non-perpendicular attachment elements. Transfer matrix for a branched structure must be considered. It is also desirable to develop a more efficient computer software based on the transfer matrix method, such as the new version of PDEMOD [12].

7. References

1. Butkovskii, A.G. and Lerner, A.Y., **"The Optimal Control of Systems with Distributed Parameters,"** Autom. Remote Control, Vol.21, pp472-477, 1960
2. Wang, P.K.C., **"Control of Distributed Parameter Systems,"** Advances in Control Systems, Vol.1, 1964
3. Brogan, W.L., **"Theory and Application of Optimal Control for Distributed Parameter,"** Automatic, Vol.4, pp.107-120, 1967
4. Pestel E.C. and Leckie, F.A., **"Matrix Methods in Elastomechanics,"** McGraw-Hill Book Company, Inc., New York, 1963

5. Horner, G.C., **"The Riccati Transfer Matrix Method,"** Ph.D. Dissertation, University of Virginia, Charlottesville, VA. May, 1975
6. Uhrig, R., **"The Transfer Matrix Method Seen as One Method of Structural Analysis Among Others,"** Journal of Sound and Vibration, Vol.4, No.2, Sept. 1966, pp136-148
7. Adams, L.R., **"Design, Development and Fabrication of a Deployable/ Retractable Truss Beam Model for Large Space Structures Application,"** NASA CR 178287, Astro Aerospace Corporation, 1987
8. Thomson, W.T., **"Theory of Vibration with Application,"** Prentice-Hall,Inc., Englewood Cliffs, New Jersey, 1972
9. Shen, J.Y., **"Maximum Likelihood Estimation for Distributed Parameter Models of Large Beam-like Structures,"** Ph.D. Dissertation, Dept. of Mechanical Engineering & Mechanics, Old Dominion University, Norfolk, VA. May 1991
10. Liu, K. and Skelton, R., **"The Q-Markov Cover and Its Application to Flexible Structure Identification,"** School of Aeronautics and Astronautics, Purdue University, 1990
11. Pappa, R.S., Schenk, A. and Noll, C., **"ERA Modal Identification Experiences with Mini-Mast,"** 2nd USAF/NASA Workshop on System Identification and Health Monitoring of Precision Space Structures, Pasadena, CA. March 1990
12. Taylor, L.W. Jr., **"PDEMOD: Computer Software for Distributed Parameter Estimation for Flexible Spacecraft Applied to NASA Mini-Mast Truss Experiment,"** 2nd USAF/NASA Workshop on System Identification and Health Monitoring of Precision Space Structures, Pasadena, CA. March 1990

Table 5.1 Comparison of Estimated Bending Frequencies (Hz.)

No.	F.E.A.	E.R.A.	D.P.A.
1	0.80	0.86	0.77
2	6.16	6.18	6.64
3	32.06	32.39	29.77
4	44.86	43.23	50.92

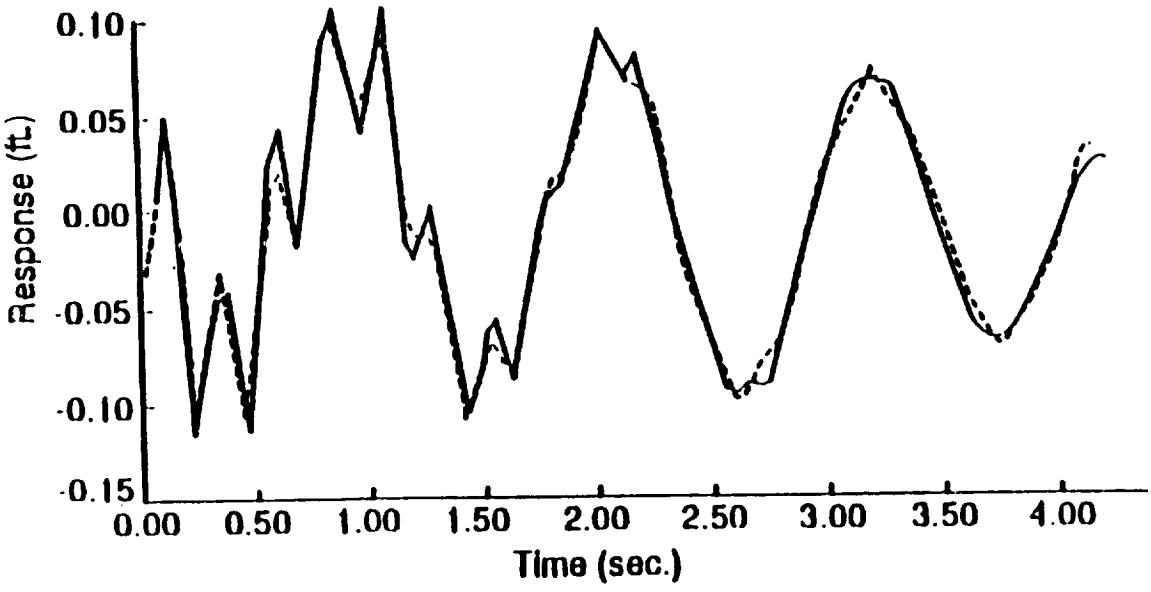

Fig. 5.1 Comparison of Reconstructed and Measured Responses